PLL Versus MAP Estimator Variance

The variance for the first-order PLL was derived in Principles of Coherent Communications, A.J. Viterbi, McGraw-Hill, equation (4.39). The definition of SNR there assumes that NO AM noise is present. The absence of AM noise improves the SNR as seen by the PLL by 3 dB since only phase noise is left to contend with the signal.

The discussions in this worksheet assume that the signal-to-noise ratio (SNR) has AM AND PM noise present. In order to be consistent, the Viterbi result therefore uses an SNR argume that is a factor of 2 larger than that presented in his text.

$$var_{PLL}(\alpha) := \frac{\pi^2}{3} + 4 \cdot \sum_{nn=1}^{99} \frac{(-1)^{nn} \cdot In(nn, \alpha \cdot 2)}{nn^2 \cdot I0(\alpha \cdot 2)}$$

In the high-SNR case, the variance of the phase error is closely approximated by one-half of the reciprocal of the SNR

$ii := 0 .. 25$

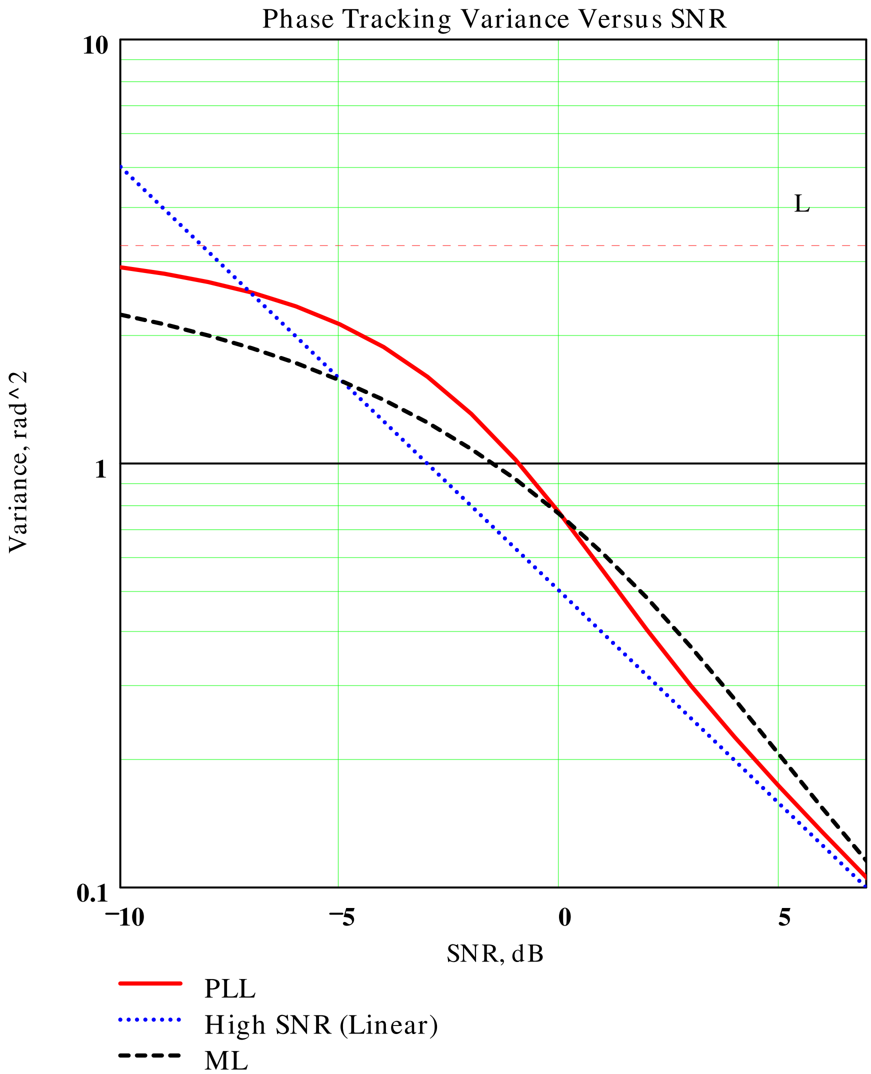
$snr_{dB_{ii}} := ii - 10$

$$var_x(\gamma) := \int_{-\pi}^{\pi} \phi^2 \cdot \frac{e^{-\gamma^2}}{2 \cdot \pi} \cdot \left[1 + \gamma \cdot \sqrt{\pi} \cdot \cos(\phi) \cdot e^{\gamma^2 \cdot \cos(\phi)^2} \cdot \left(1 + erf(\gamma \cdot \cos(\phi))\right) \right] d\phi$$

$$hsnr1_{ii} := \frac{10^{-0.1 \cdot snr_{dB_{ii}}}}{2}$$

$$trk_{PLL_{ii}} := var_{PLL}\left(10^{0.1 \cdot snr_{dB_{ii}}}\right) \qquad trk_{x_{ii}} := var_x\left(\sqrt{10^{0.1 \cdot snr_{dB_{ii}}}}\right)$$

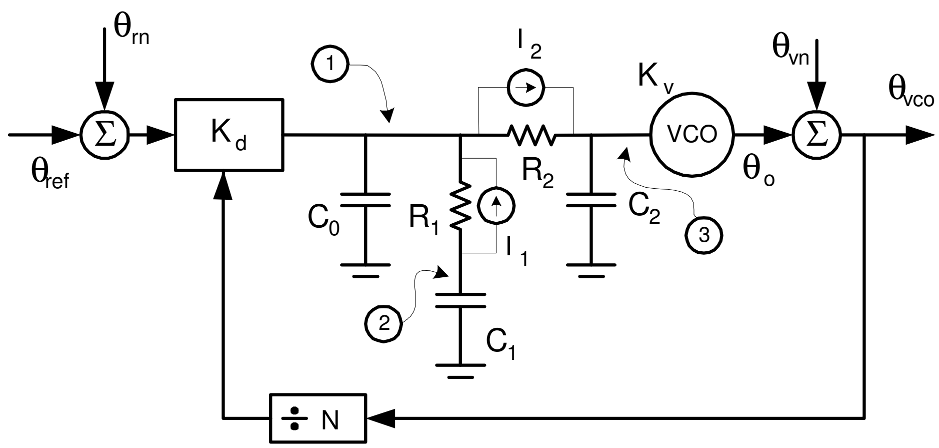
$$L := \frac{\pi^2}{3}$$



Phase Tracking Variance Versus SNR
10
1
0.1
Variance, rad^2
L
−10
−5
0
5
SNR, dB
PLL
High SNR (Linear)
ML

Type-2 4th-Order PLL

J.A. Crawford
24 March 2004



$K_d := \frac{0.001}{2\cdot\pi}$ $\quad K_v := 2\cdot\pi\cdot 10^7$

$F_n := 2.5\cdot 10^3$ $\quad \omega_n := F_n\cdot 2\cdot\pi$

$N := 9000$

$\zeta := 0.75$

$F_{ref} := 10^5$

$C_1 := \left(\frac{\omega_n{}^2\cdot N}{K_d\cdot K_v}\right)^{-1}$

Arbitrary Choices

$R_1 := \frac{2\cdot\zeta}{\omega_n\cdot C_1}$

$C_0 := 0.01\cdot C_1$

$R_2 := 10\cdot R_1$

$C_2 := \left(5\cdot\omega_n\cdot R_2\right)^{-1}$ $\quad C_0 = 4.503\times 10^{-11}$ $\quad R_1 = 2.121\times 10^4$

$G_1 := R_1{}^{-1}$ $\quad C_1 = 4.503\times 10^{-9}$ $\quad R_2 = 2.121\times 10^5$

$G_2 := R_2{}^{-1}$ $\quad C_2 = 6.004\times 10^{-11}$

$$SV(s,p1,p2,p3,p4) := \begin{pmatrix} s\cdot C_0 + G_1 + G_2 & -G_1 & -G_2 & \frac{K_d}{N} \\ -G_1 & G_1 + s\cdot C_1 & 0 & 0 \\ -G_2 & 0 & G_2 + s\cdot C_2 & 0 \\ 0 & 0 & \frac{-K_v}{s} & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} p1 \\ p2 \\ p3 \\ p4 \end{pmatrix}$$

Reference Noise at Output

$$jx := \sqrt{-1}$$

$$RefNoise(f,\theta_{rn}) := \left| \begin{array}{l} S \leftarrow SV(jx\cdot 2\cdot \pi \cdot f, K_d\cdot \theta_{rn}, 0, 0, 0) \\ NoiseOut \leftarrow 20\cdot \log(|S_3|) \end{array} \right.$$

Self-VCO Noise at Output

$$VCONoise(f,L_{vco}) := \left| \begin{array}{l} S \leftarrow SV(jx\cdot 2\cdot \pi \cdot f, 0, 0, 0, L_{vco}) \\ NoiseOut \leftarrow 20\cdot \log(|S_3|) \end{array} \right.$$

Leeson Noise Model for VCO

$F_o := N\cdot F_{ref}$ VCO Center Frequency

$P_{out} := 0.001$ Output power, Watts

$k := 1.38\cdot 10^{-23}$ Boltzman constant

$T := 290$ Temperature

$F := 2$ Noise Factor

$$L_{dBc}(f,Q_L) := 10\cdot \log\left[\frac{F\cdot k\cdot T}{2\cdot P_{out}}\cdot \left[1 + \left(\frac{F_o}{2\cdot Q_L\cdot f}\right)^2\right]\right]$$

Phase Detector Noise Floor

$$L_{pd\_dBc} := -160$$

$N_{pts} := 75 \qquad nn := 0 .. N_{pts} - 1$

$$fsw_{nn} := 10^{\frac{nn}{N_{pts}} \cdot 6}$$

$$N1_{nn} := RefNoise\left(fsw_{nn}, 10^{0.05 \cdot L_{pd\_dBc}}\right) \qquad N2_{nn} := VCONoise\left(fsw_{nn}, 10^{0.05 \cdot L_{dBc}(fsw_{nn}, 5)}\right)$$

Resistor Noise Contributions

$$R1Noise(f) := \left| \begin{array}{l} S \leftarrow SV\left(jx \cdot 2 \cdot \pi \cdot f, \sqrt{\frac{4 \cdot k \cdot T}{R_1}}, -\sqrt{\frac{4 \cdot k \cdot T}{R_1}}, 0, 0\right) \\ NoiseOut \leftarrow 20 \cdot \log\left(\left|S_3\right|\right) \end{array} \right.$$

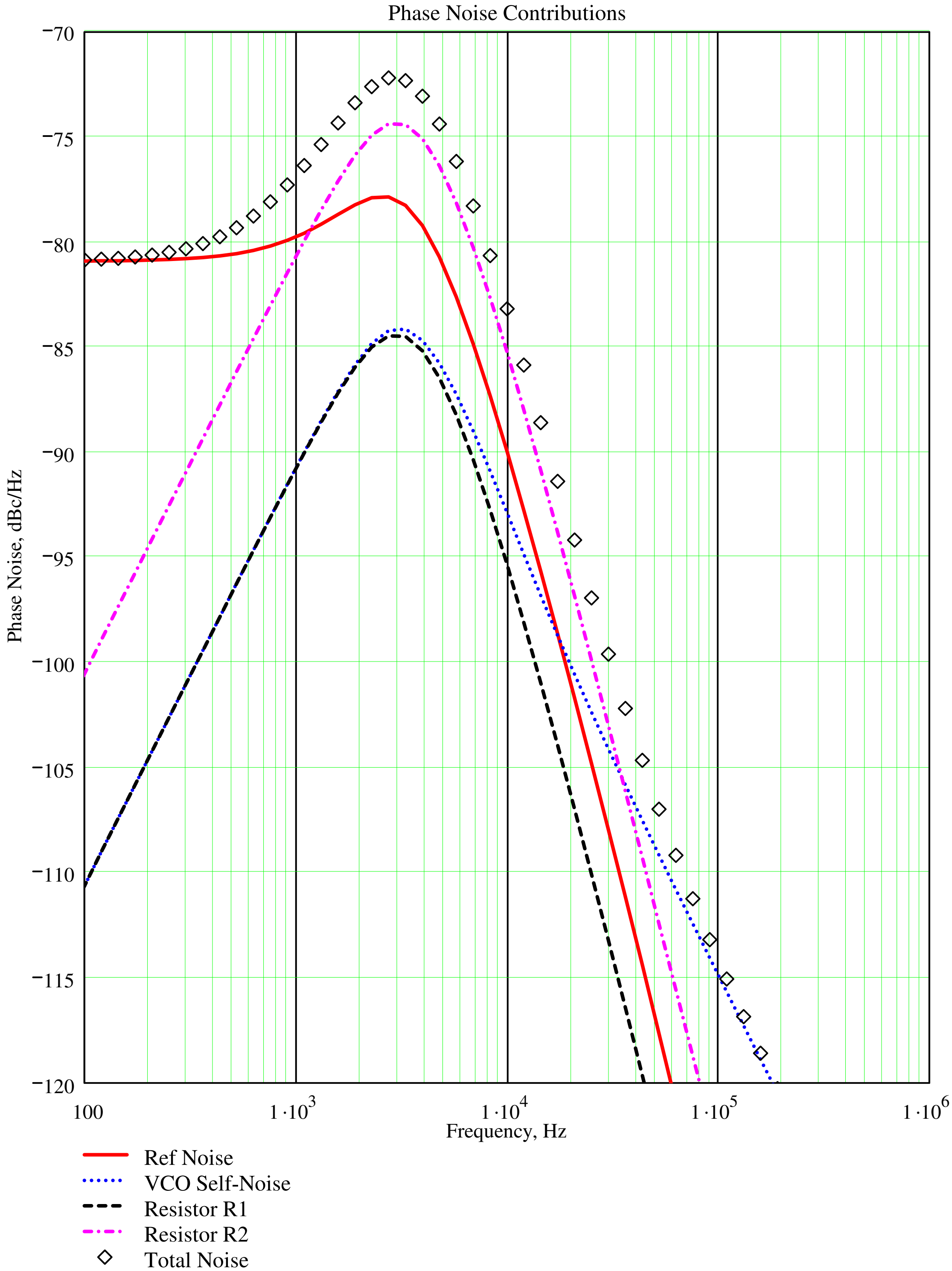
$$N3_{nn} := R1Noise\left(fsw_{nn}\right)$$

Resistor R2 Noise

$$R2Noise(f) := \left| \begin{array}{l} S \leftarrow SV\left(jx \cdot 2 \cdot \pi \cdot f, -\sqrt{\frac{4 \cdot k \cdot T}{R_2}}, 0, \sqrt{\frac{4 \cdot k \cdot T}{R_2}}, 0\right) \\ NoiseOut \leftarrow 20 \cdot \log\left(\left|S_3\right|\right) \end{array} \right.$$

$$N4_{nn} := R2Noise\left(fsw_{nn}\right)$$

$$NT_{nn} := 10 \cdot \log\left(10^{0.1 \cdot N1_{nn}} + 10^{0.1 \cdot N2_{nn}} + 10^{0.1 \cdot N3_{nn}} + 10^{0.1 \cdot N4_{nn}}\right)$$


Phase Noise Contributions
Phase Noise, dBc/Hz
−70
−75
−80
−85
−90
−95
−100
−105
−110
−115
−120
100
1·10^3
1·10^4
1·10^5
1·10^6
Frequency, Hz
Ref Noise
VCO Self-Noise
Resistor R1
Resistor R2
Total Noise

VCO Self-Noise to Output Transfer Function

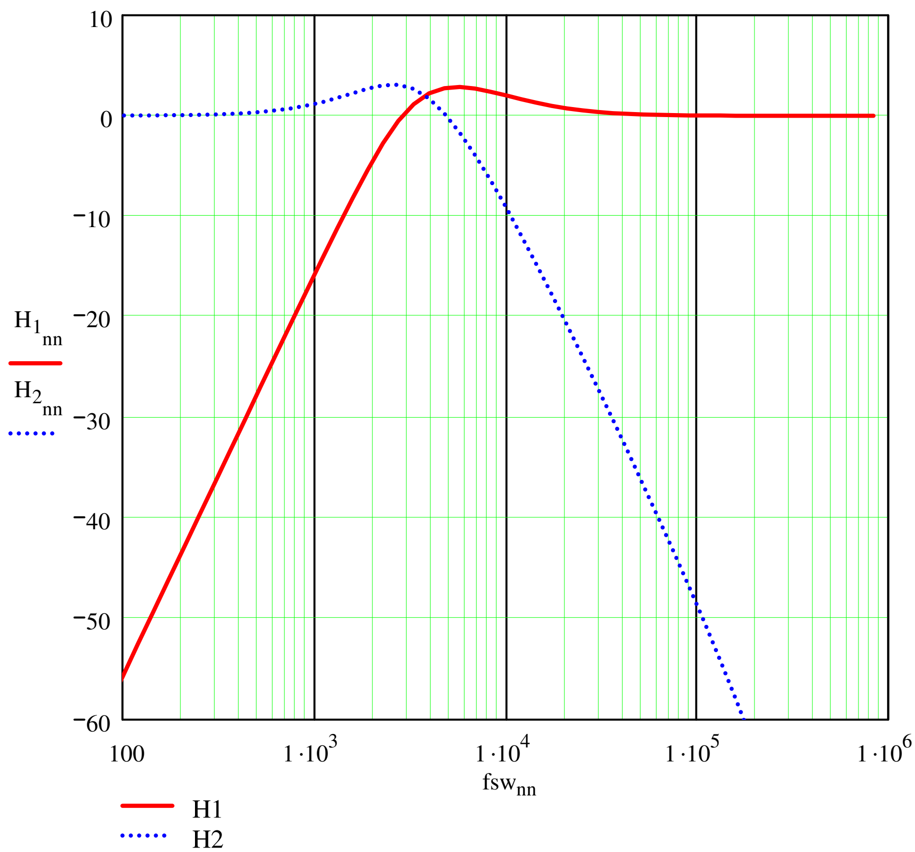
$$H1(f) := \left| \begin{array}{l} S \leftarrow SV(jx \cdot 2 \cdot \pi \cdot f, 0, 0, 0, 1) \\ G \leftarrow 20 \cdot \log(|S_3|) \end{array} \right.$$

$$H_{1_{nn}} := H1(fsw_{nn})$$

Reference Noise to Output Transfer Function

$$H2(f) := \left| \begin{array}{l} S \leftarrow SV(jx \cdot 2 \cdot \pi \cdot f, K_d, 0, 0, 0) \\ G \leftarrow 20 \cdot \log(|S_3|) \end{array} \right.$$

$$H_{2_{nn}} := H2(fsw_{nn}) - 20 \cdot \log(N)$$



Use FFT to Compute Transient Response

A step-change in output frequency can be equivalenced to a phase ramp at the phase detector

$FFT_{size} := \frac{4096}{4}$

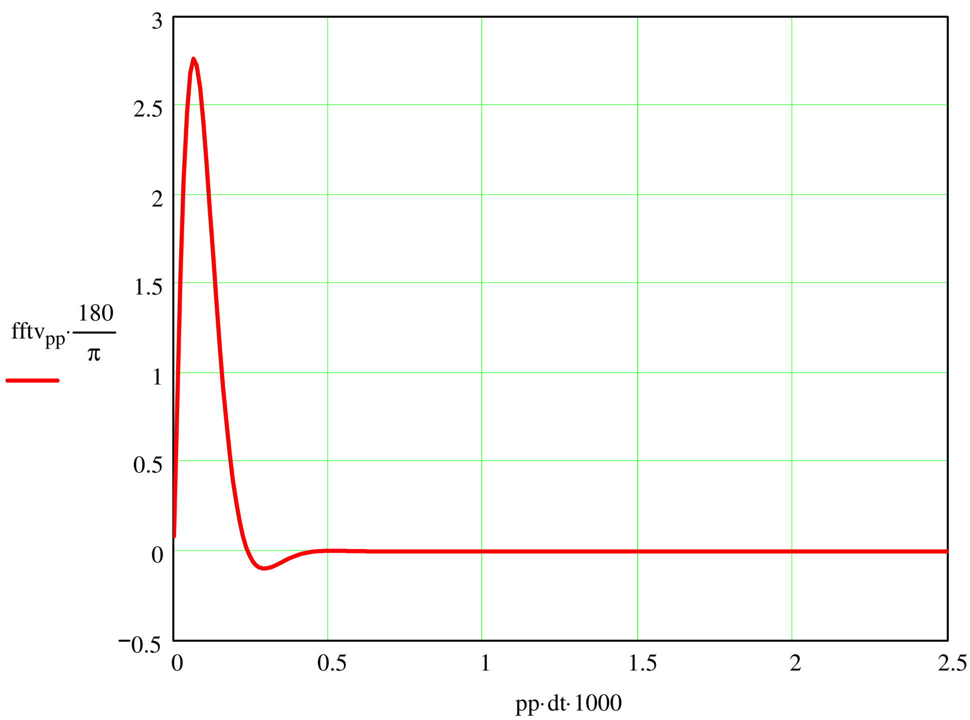
$dt := (8 \cdot \zeta \cdot \omega_n)^{-1}$ $df := (FFT_{size} \cdot dt \cdot 2)^{-1}$ $df = 46.019$ $Fstep := 10^4$

$dt^{-1} = 9.425 \times 10^4$

$ss := 1 .. FFT_{size}$

$FFTv_{ss} := SV\left[jx \cdot 2 \cdot \pi \cdot df \cdot ss, 0, 0, 0, \frac{2 \cdot \pi \cdot Fstep}{(jx \cdot 2 \cdot \pi \cdot df \cdot ss)^2} \right]_3$ $FFTv_0 := Re(FFTv_1)$

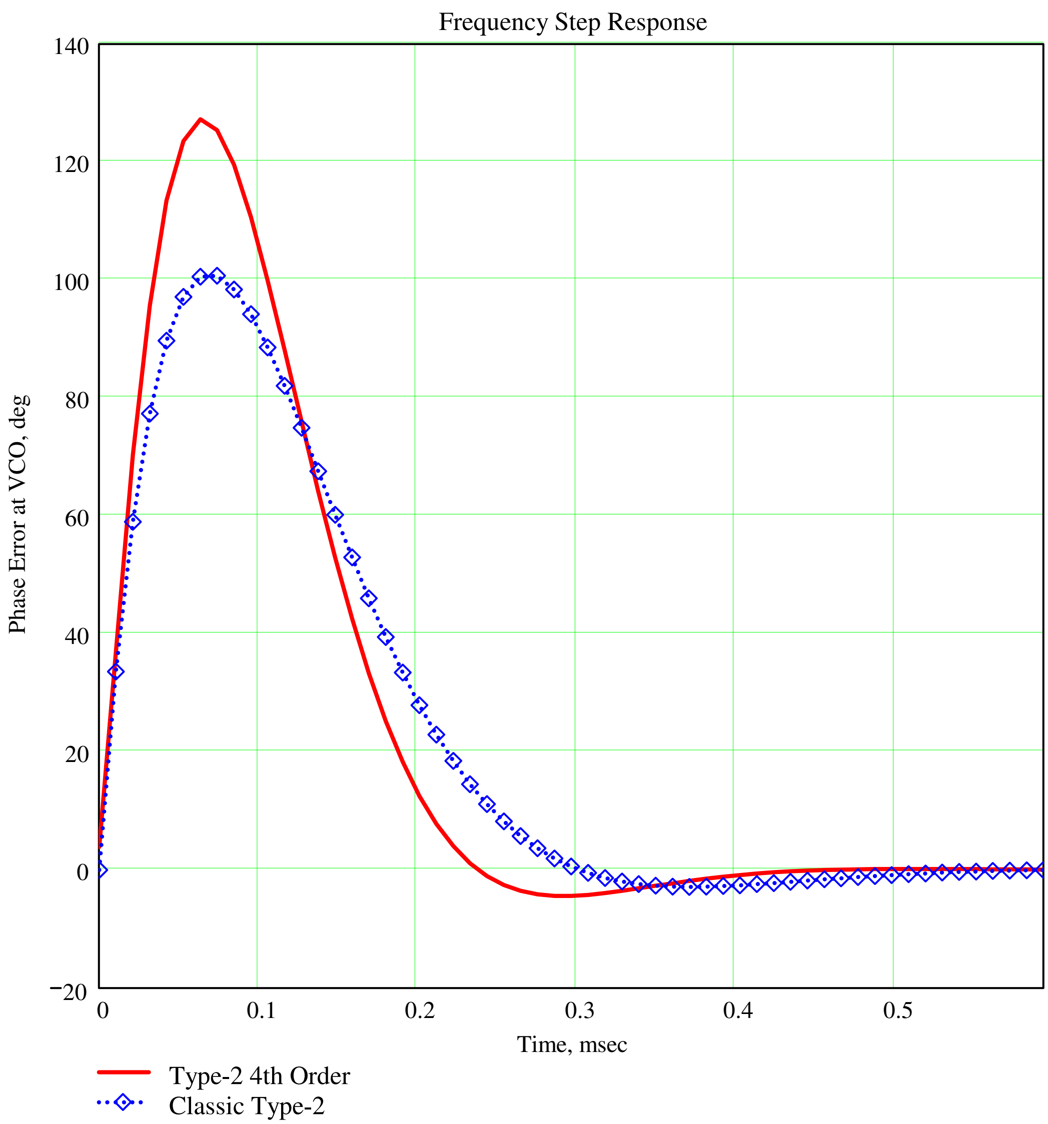
$fftv := IFFT(FFTv)$ $pp := 0 .. rows(fftv) - 1$ $rows(fftv) = 2.048 \times 10^3$



$\gamma_s := \frac{dt^{-1}}{2 \cdot FFT_{size}}$

Closed-Form Result for Type-2 System

$$\theta_{pd}(t) := \frac{2\cdot\pi\cdot Fstep}{\omega_n}\cdot\frac{e^{-\zeta\cdot\omega_n\cdot t}}{\sqrt{1-\zeta^2}}\cdot\sin\left(\omega_n\cdot\sqrt{1-\zeta^2}\cdot t\right)$$



BER for QAM

$$\mathrm{bits}(x) := \mathrm{floor}\left(\frac{\log(x)}{\log(2)} + 0.1\right) \qquad \mathrm{erfc}(x) := 1 - \mathrm{erf}(x)$$

16 QAM

$$\mathrm{bits}(16) = 4$$

BPSK case added at the end
26 October 2001

$$Q(x) := \frac{1}{2}\cdot\left(1 - \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right)$$

$$P_{b16QAM}(EsNo) := \frac{1}{2}\cdot Q\left(\sqrt{\frac{EsNo}{5}}\right) + \frac{1}{2}\cdot Q\left(3\cdot\sqrt{\frac{EsNo}{5}}\right)$$

From Modern Quadrature Amplitude Modulation by W.T. Webb, L. Hanzo, Pentech Press, 1994, page 161

$$SNR_{bit\_dB} := 15$$

EsNo is the AVERAGE symbol to noise ratio

$$P_{b16QAM}\left[10^{0.1\cdot\left(SNR_{bit\_dB}\right)}\cdot \mathrm{bits}(16)\right] = 1.228\times 10^{-7}$$ **Bit Error Rate**

From Digital Communications, Proakis, Wiley, 2nd Ed., 1989, page 282

Symbol Error Rate

$$P_s(\gamma_b, M, k) := 2\cdot\left(1 - \frac{1}{\sqrt{M}}\right)\cdot \mathrm{erfc}\left[\sqrt{\frac{3\cdot k\cdot\gamma_b}{2\cdot(M-1)}}\right]\cdot\left[1 - \frac{1}{2}\cdot\left(1 - \frac{1}{\sqrt{M}}\right)\cdot \mathrm{erfc}\left[\sqrt{\frac{3\cdot k\cdot\gamma_b}{2\cdot(M-1)}}\right]\right] \qquad \text{(equ. 4.2.144)}$$

Here, γ_b is the average SNR per bit

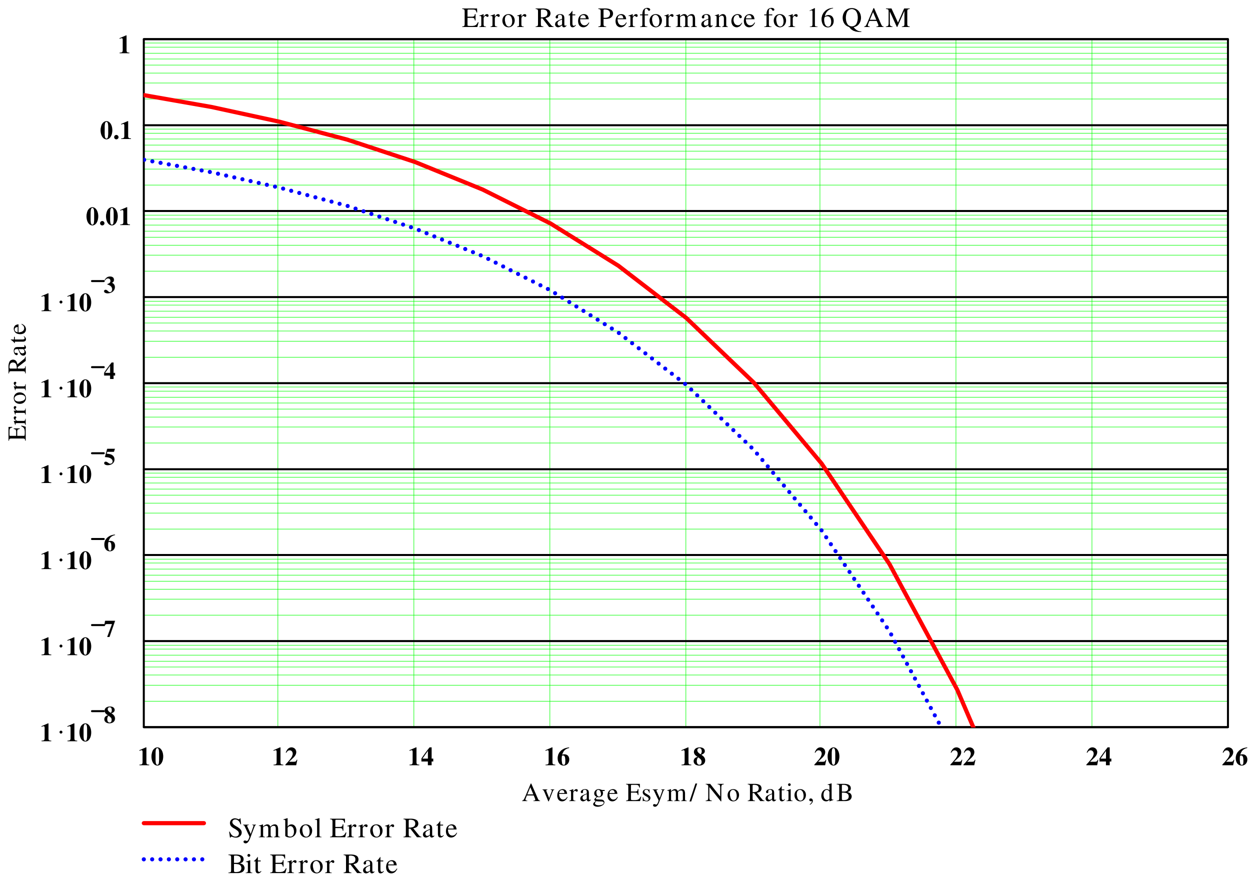
k is the number of bits per symbol
M is the total number of points in the square QAM constellation (256)

$$P_s\left(10^{0.1\cdot SNR_{bit\_dB}}, 256, 8\right) = 0.15215195 \qquad \frac{3}{2}\cdot \mathrm{erfc}\left(\sqrt{\frac{2}{5}\cdot 10^{0.1\cdot SNR_{bit\_dB}}}\right) = 7.367\times 10^{-7}$$

$kb := 0 .. 15$

$EsNo_{dB_{kb}} := 10 + kb$ $EbNo_{dB_{kb}} := EsNo_{dB_{kb}} - 10 \cdot \log(bits(16))$

$$ps_{kb} := P_s\left(\frac{10^{0.1 \cdot EsNo_{dB_{kb}}}}{4}, 16, 4\right) \qquad pb_{kb} := P_{b16QAM}\left(10^{0.1 \cdot EsNo_{dB_{kb}}}\right)$$



Derived Phase Noise Impact for Square QAM Constellations

$$\text{AvEnergyPerSymbol}(\text{CSize}) := \left| \begin{array}{l} \text{Nbits} \leftarrow \text{floor}\left(\dfrac{\log(\text{CSize})}{\log(2)} + 0.1\right) \\ \text{RailLevels} \leftarrow 2^{0.5\cdot\text{Nbits}} \\ \text{sum} \leftarrow 0 \\ \text{for } ii \in 0 .. \dfrac{\text{RailLevels}}{2} - 1 \\ \quad \text{for } jj \in 0 .. \dfrac{\text{RailLevels}}{2} - 1 \\ \quad\quad \text{sum} \leftarrow \text{sum} + \left[\left(\dfrac{2\cdot ii + 1}{2}\right)^2 + \left(\dfrac{2\cdot jj + 1}{2}\right)^2\right] \\ \text{sum} \leftarrow \dfrac{4\cdot\text{sum}}{\text{CSize}} \end{array} \right.$$

$$\text{AvEnergyPerSymbol}(256) = 42.5$$

Assumes square QAM constellation.

Adds up power for all of the constellation points in the upper right-hand quadrant, multiplies this sum by 4 to get the total for the entire constelllation, and then divides by the total number of constellation points.

Rectangular spacing between constellation points assumed to be d= 1.

$$P_{sym}(\theta_n, SNRdB_{bit}, CSize) := \left| \begin{array}{l} Nbits \leftarrow floor\left(\frac{\log(CSize)}{\log(2)} + 0.1\right) \\ RailLevels \leftarrow 2^{\frac{Nbits}{2}} \\ snr_{bit} \leftarrow 10^{0.1 \cdot SNRdB_{bit}} \\ E \leftarrow AvEnergyPerSymbol(CSize) \\ \sigma \leftarrow \sqrt{\frac{E}{2 \cdot snr_{bit} \cdot Nbits}} \\ sum \leftarrow \displaystyle\sum_{k=0}^{\frac{RailLevels}{2}-1} erfc\left[\frac{\frac{1}{2} - \frac{(2 \cdot k + 1)}{2} \cdot \sin(\theta_n)}{\sigma \cdot \sqrt{2}}\right] \\ sum \leftarrow sum + \displaystyle\sum_{k=0}^{\frac{RailLevels}{2}-1} erfc\left[\frac{\frac{1}{2} + \frac{(2 \cdot k + 1)}{2} \cdot \sin(\theta_n)}{\sigma \cdot \sqrt{2}}\right] \\ sum \leftarrow sum \cdot 2 \cdot \left(\frac{RailLevels - 1}{RailLevels^2}\right) \end{array} \right.$$

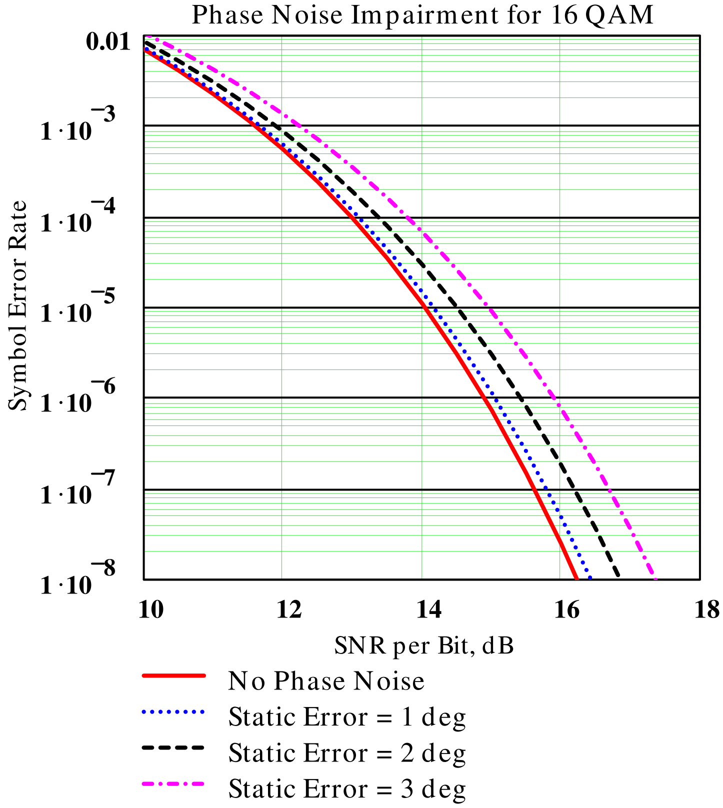
$pp := 0..50$

$SNR_{bitdB_{pp}} := 2 + pp \cdot 0.5$

16 QAM Situation

$y1_{pp} := P_{sym}\left(0, SNR_{bitdB_{pp}}, 16\right)$ $y2_{pp} := P_{sym}\left(\frac{1}{180}\cdot\pi, SNR_{bitdB_{pp}}, 16\right)$

$y3_{pp} := P_{sym}\left(\frac{2}{180}\cdot\pi, SNR_{bitdB_{pp}}, 16\right)$ $y4_{pp} := P_{sym}\left(\frac{3}{180}\cdot\pi, SNR_{bitdB_{pp}}, 16\right)$

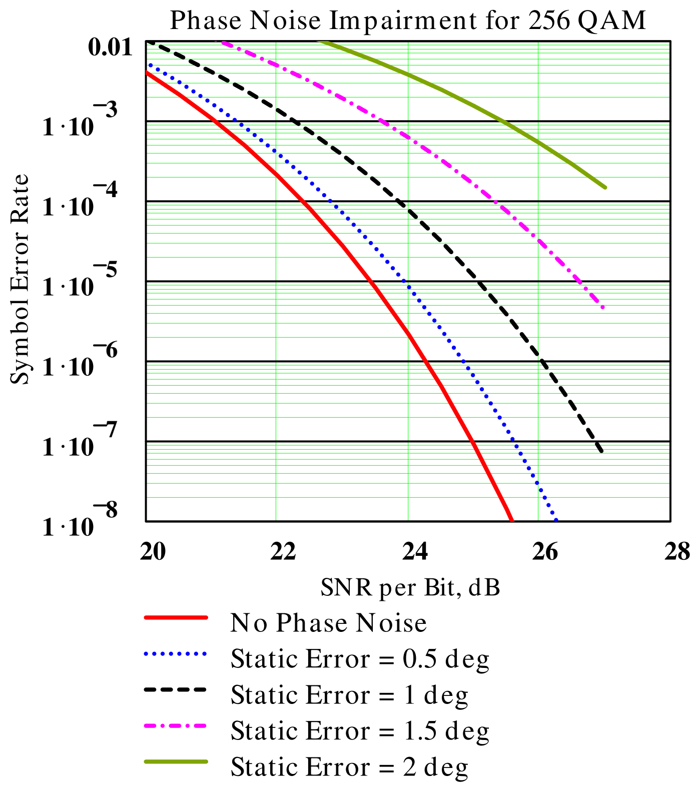


256 QAM Situation

$$y1_{pp} := P_{sym}\left(0, SNR_{bitdB_{pp}}, 256\right)$$

$$y2_{pp} := P_{sym}\left(\frac{0.5}{180}\cdot\pi, SNR_{bitdB_{pp}}, 256\right) \qquad y4_{pp} := P_{sym}\left(\frac{1.5}{180}\cdot\pi, SNR_{bitdB_{pp}}, 256\right)$$

$$y3_{pp} := P_{sym}\left(\frac{1}{180}\cdot\pi, SNR_{bitdB_{pp}}, 256\right) \qquad y5_{pp} := P_{sym}\left(\frac{2.0}{180}\cdot\pi, SNR_{bitdB_{pp}}, 256\right)$$



64 QAM Situation

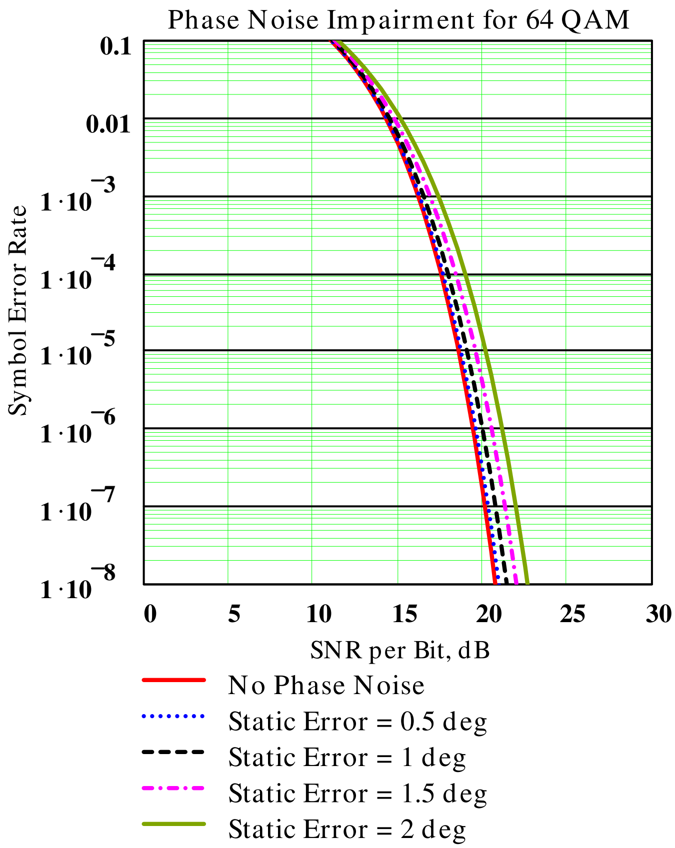
$$y1_{pp} := P_{sym}\left(\mathbf{0}, SNR_{bitdB_{pp}}, \mathbf{64}\right)$$

$$y2_{pp} := P_{sym}\left(\frac{\mathbf{0.5}}{\mathbf{180}}\cdot\pi, SNR_{bitdB_{pp}}, \mathbf{64}\right)$$

$$y3_{pp} := P_{sym}\left(\frac{\mathbf{1}}{\mathbf{180}}\cdot\pi, SNR_{bitdB_{pp}}, \mathbf{64}\right)$$

$$y4_{pp} := P_{sym}\left(\frac{\mathbf{1.5}}{\mathbf{180}}\cdot\pi, SNR_{bitdB_{pp}}, \mathbf{64}\right)$$

$$y5_{pp} := P_{sym}\left(\frac{\mathbf{2.0}}{\mathbf{180}}\cdot\pi, SNR_{bitdB_{pp}}, \mathbf{64}\right)$$



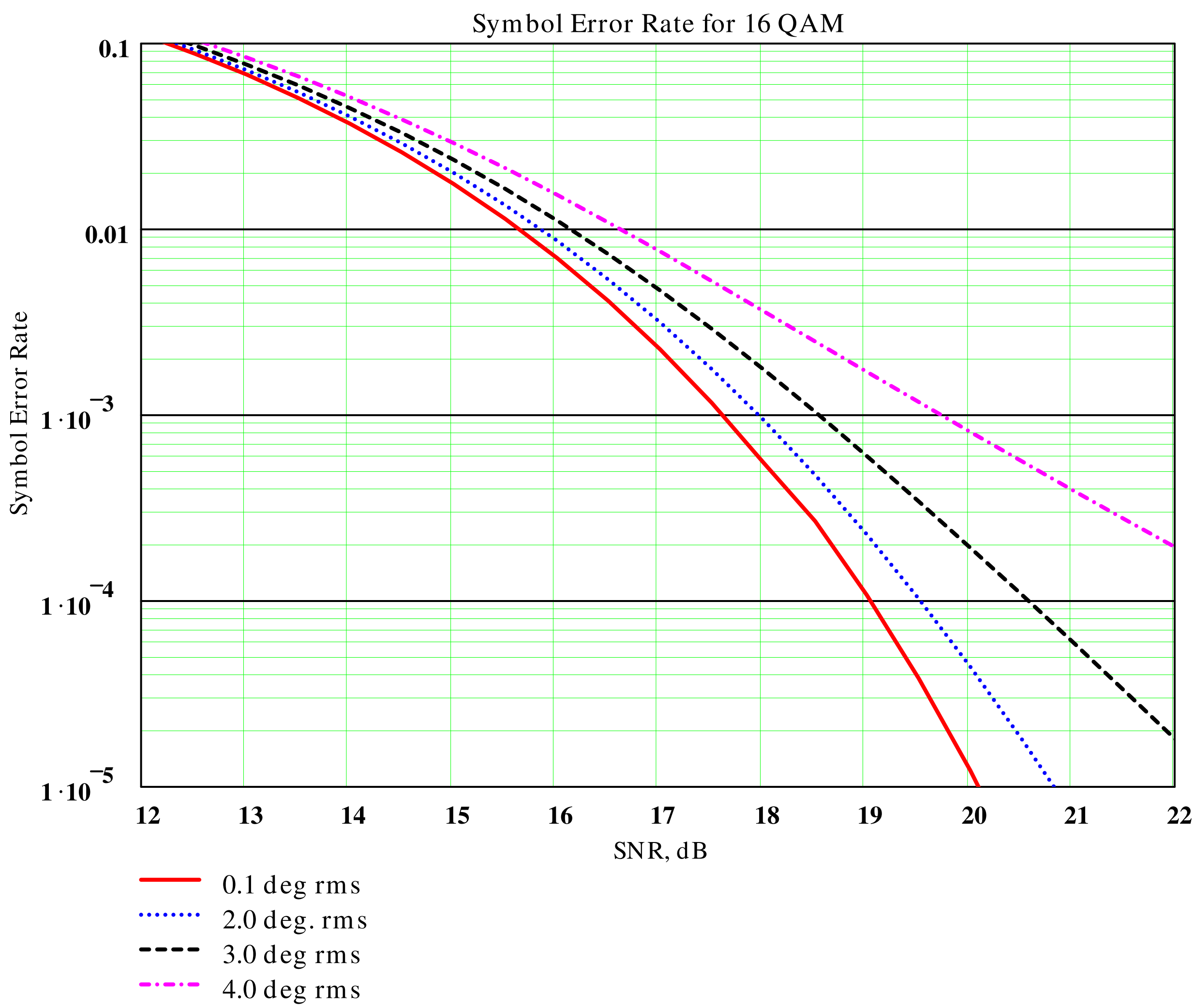
Now, assume the Tikhonov PDF for the phase error: 16 QAM Case

$$p_\theta(\theta,\sigma_\theta) := \frac{1}{\sqrt{2\cdot\pi}\cdot\sigma_\theta}\cdot e^{\left(\frac{\cos(\theta)-1}{\sigma_\theta^{2}}\right)}$$

$$PsWithNoise16(SNR_{bitdB},\theta_{degrms}) := \int_{-0.5}^{0.5} p_\theta\left(\theta,\frac{\theta_{degrms}}{180}\cdot\pi\right)\cdot P_{sym}(\theta, SNR_{bitdB}, 16)\, d\theta$$

$ph1_{pp} := PsWithNoise16\left(SNR_{bitdB_{pp}}, 0.1\right)$ $ph2_{pp} := PsWithNoise16\left(SNR_{bitdB_{pp}}, 2\right)$

$ph3_{pp} := PsWithNoise16\left(SNR_{bitdB_{pp}}, 3\right)$ $ph4_{pp} := PsWithNoise16\left(SNR_{bitdB_{pp}}, 4\right)$



Now, assume the Tikhonov PDF for the phase error: 64 QAM Case

$$PsWithNoise64\left(SNR_{bitdB},\theta_{degrms}\right) := \int_{-0.5}^{0.5} p_{\theta}\left(\theta,\frac{\theta_{degrms}}{180}\cdot\pi\right)\cdot P_{sym}\left(\theta,SNR_{bitdB},\mathbf{64}\right)d\theta$$

$$ph1_{pp} := PsWithNoise64\left(SNR_{bitdB_{pp}},\mathbf{0.1}\right)$$

$$ph2_{pp} := PsWithNoise64\left(SNR_{bitdB_{pp}},\mathbf{1.0}\right)$$

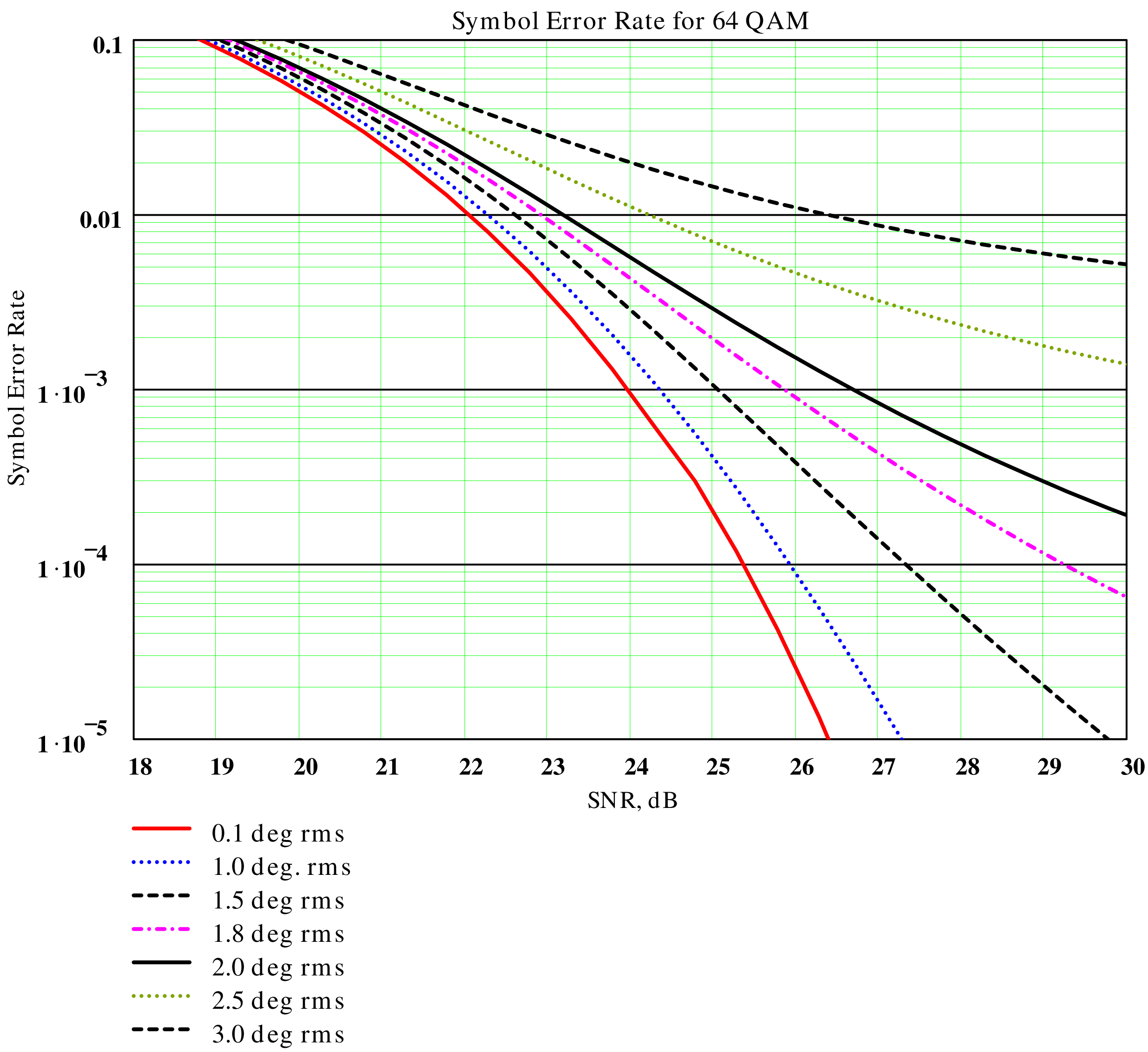
$$ph3_{pp} := PsWithNoise64\left(SNR_{bitdB_{pp}},\mathbf{1.5}\right)$$

$$ph4_{pp} := PsWithNoise64\left(SNR_{bitdB_{pp}},\mathbf{1.8}\right)$$

$$ph5_{pp} := PsWithNoise64\left(SNR_{bitdB_{pp}},\mathbf{2}\right)$$

$$ph6_{pp} := PsWithNoise64\left(SNR_{bitdB_{pp}},\mathbf{2.5}\right)$$

$$ph7_{pp} := PsWithNoise64\left(SNR_{bitdB_{pp}},\mathbf{3}\right)$$



Now, assume the Tikhonov PDF for the phase error: 256 QAM Case

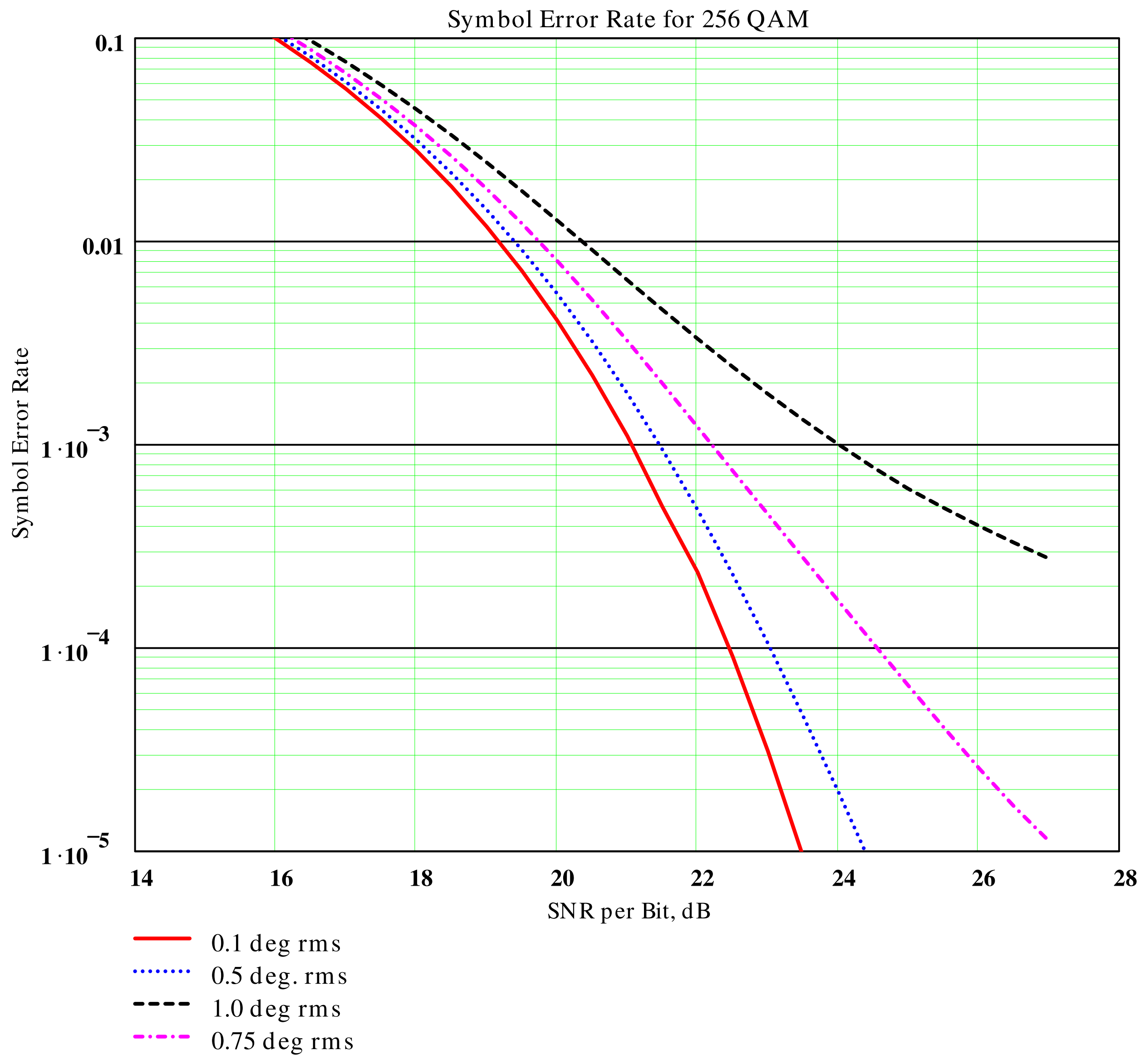
$$PsWithNoise256\left(SNR_{bitdB}, \theta_{degrms}\right) := \int_{-0.5}^{0.5} p_{\theta}\left(\theta, \frac{\theta_{degrms}}{180}\cdot\pi\right)\cdot P_{sym}\left(\theta, SNR_{bitdB}, 256\right) d\theta$$

$$ph1_{pp} := PsWithNoise256\left(SNR_{bitdB_{pp}}, 0.1\right)$$

$$ph2_{pp} := PsWithNoise256\left(SNR_{bitdB_{pp}}, 0.5\right)$$

$$ph3_{pp} := PsWithNoise256\left(SNR_{bitdB_{pp}}, 1\right)$$

$$ph4_{pp} := PsWithNoise256\left(SNR_{bitdB_{pp}}, 0.75\right)$$



$\mathrm{erf}(1) = 0.8427$

$$\int_{-1}^{1} \frac{e^{(-u^2)}}{\sqrt{\pi}}\,du = 0.8427$$

Rayleigh Fading Compensation Method for 16QAM in Digital Land Mobile Radio Channels", IEEE 1989, S. Sampei, T. Sunaga

BER for Gray-coded 16-QAM with coherent detection

$$P_b(\gamma_o) := \frac{3}{8}\cdot \mathrm{erfc}\left(\sqrt{0.4\cdot\gamma_o}\right) - \frac{9}{64}\cdot \mathrm{erfc}\left(\sqrt{0.4\cdot\gamma_o}\right)^2$$

Rayleigh Fading Compensation for QAM in Land Mobile Radio Communications", IEE Trans. Vehicular Tech., May 1993, S. Sampei, T. Sunaga

BER for 64-QAM

$$P_b(\gamma_o) := \frac{4}{27}\cdot \mathrm{erfc}\left(\frac{1}{7}\cdot\gamma_o\right) - \frac{49}{384}\cdot \mathrm{erfc}\left(\sqrt{\frac{1}{7}\cdot\gamma_o}\right)^2$$

BER for 256 QAM

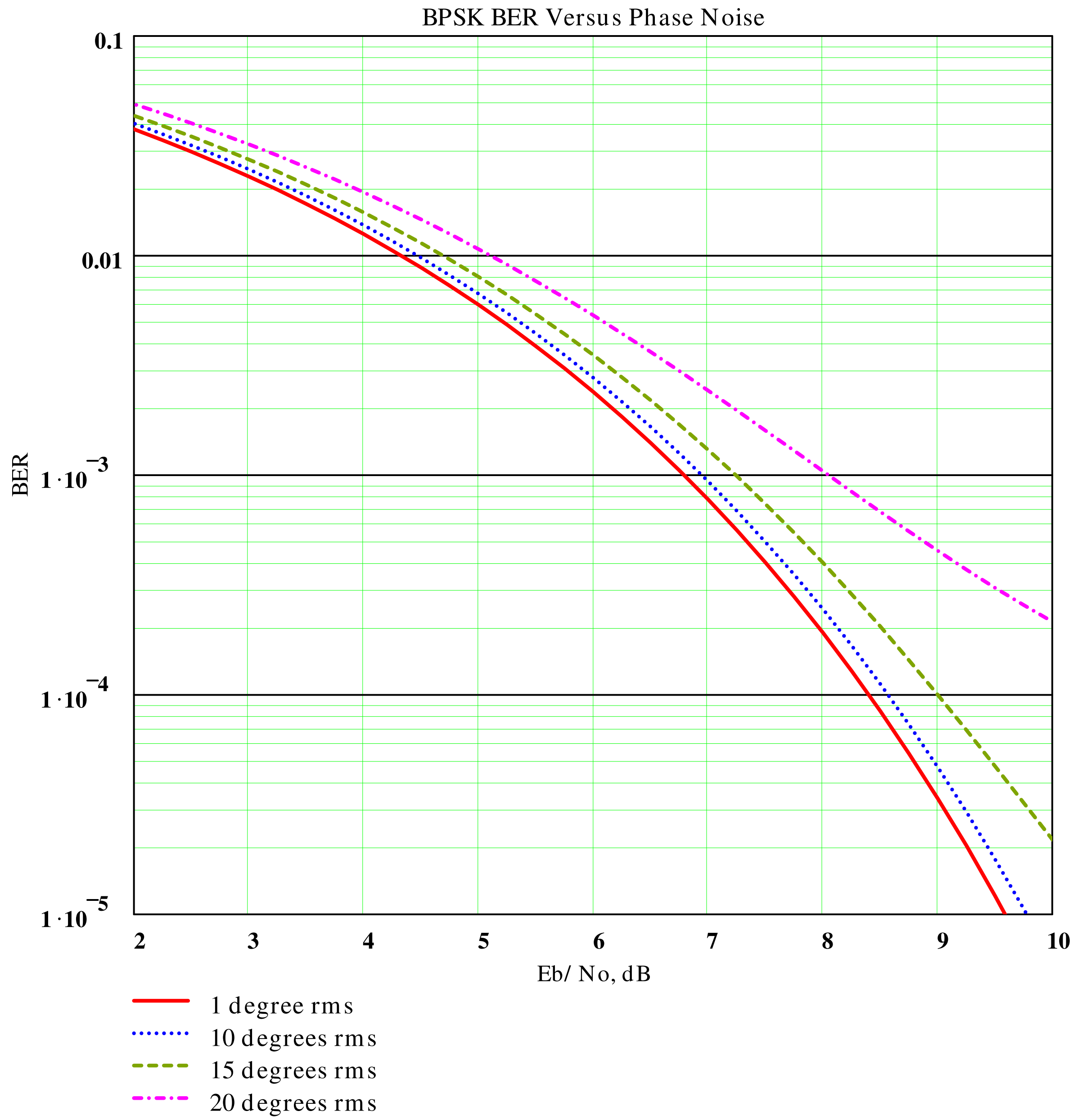
$$P_b(\gamma_o) := \frac{15}{64}\cdot \mathrm{erfc}\left(\sqrt{\frac{4}{85}\cdot\gamma_o}\right) - \frac{225}{2048}\cdot \mathrm{erfc}\left(\sqrt{\frac{4}{85}\cdot\gamma_o}\right)^2$$

Look at BER with BPSK

$kp := 0..40 \qquad x_{kp} := kp \cdot 0.25$

$$Pb(EbNo_{dB}, \sigma p) := 2 \cdot \int_0^{8 \cdot \sigma p} \frac{1}{2} \cdot erfc\left(10^{0.05 \cdot EbNo_{dB}} \cdot \cos(\theta)\right) \cdot \frac{1}{\sqrt{2 \cdot \pi} \cdot \sigma p} \cdot e^{-0.5 \cdot \left(\frac{\theta}{\sigma p}\right)^2} d\theta$$

$y1_{kp} := Pb\left(x_{kp}, \frac{1}{180} \cdot \pi\right) \qquad y2_{kp} := Pb\left(x_{kp}, \frac{10}{180} \cdot \pi\right) \qquad y3_{kp} := Pb\left(x_{kp}, \frac{15}{180} \cdot \pi\right) \qquad y4_{kp} := Pb\left(x_{kp}, \frac{20}{180} \cdot \pi\right)$



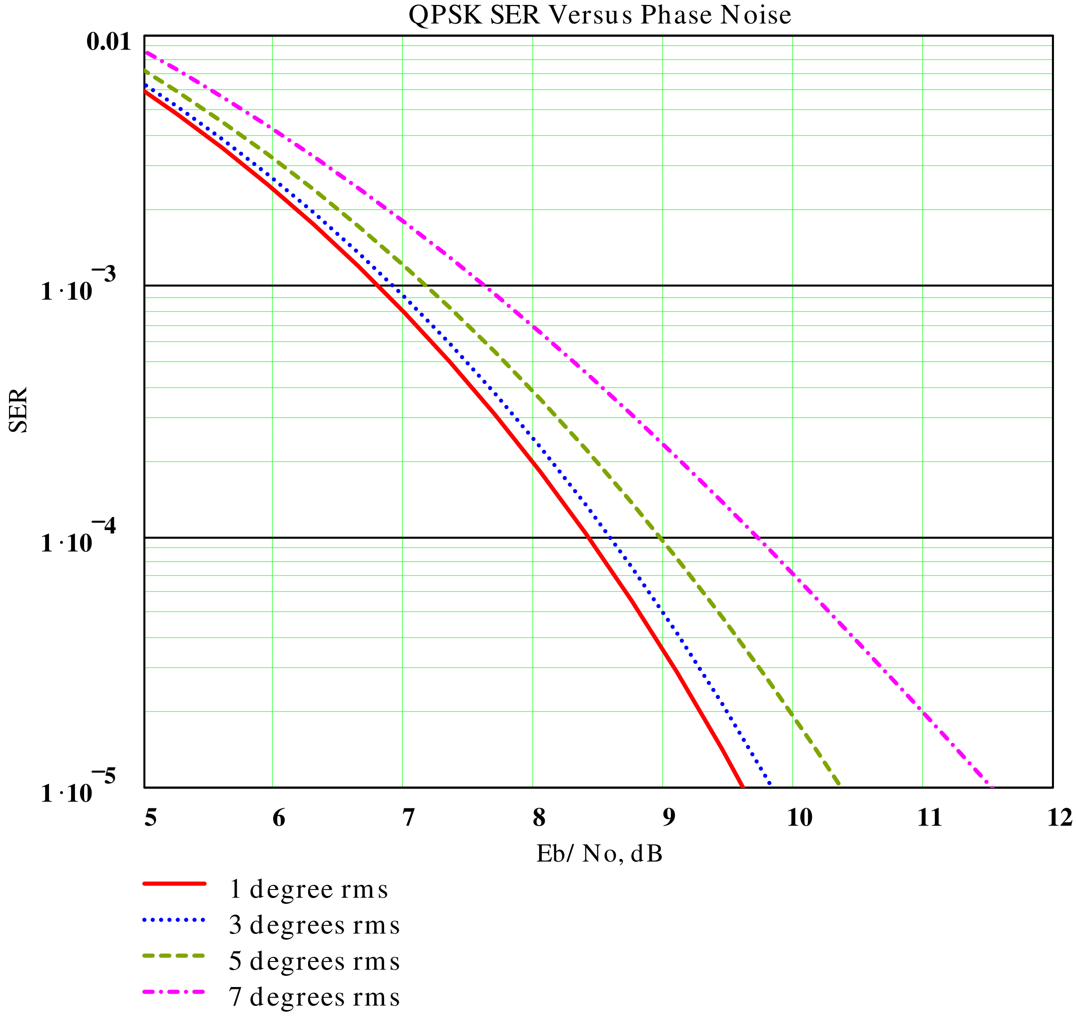
Look at BER with QPSK

$kp := 0..40 \qquad x_{kp} := kp \cdot 0.35$

$$Pbq(EbNo_{dB}, \sigma p) := \int_{0}^{8\cdot\sigma p} \frac{1}{2}\cdot \mathrm{erfc}\left[10^{0.05\cdot EbNo_{dB}}\cdot(\cos(\theta) - \sin(\theta))\right]\cdot\frac{1}{\sqrt{2\cdot\pi}\cdot\sigma p}\cdot e^{-0.5\cdot\left(\frac{\theta}{\sigma p}\right)^2}\,d\theta \;...$$

$$+ \int_{0}^{8\cdot\sigma p} \frac{1}{2}\cdot \mathrm{erfc}\left[10^{0.05\cdot EbNo_{dB}}\cdot(\cos(\theta) + \sin(\theta))\right]\cdot\frac{1}{\sqrt{2\cdot\pi}\cdot\sigma p}\cdot e^{-0.5\cdot\left(\frac{\theta}{\sigma p}\right)^2}\,d\theta$$

$y1_{kp} := Pbq\left(x_{kp}, \frac{1}{180}\cdot\pi\right) \qquad y2_{kp} := Pbq\left(x_{kp}, \frac{3}{180}\cdot\pi\right) \qquad y3_{kp} := Pbq\left(x_{kp}, \frac{5}{180}\cdot\pi\right) \qquad y4_{kp} := Pbq\left(x_{kp}, \frac{7}{180}\cdot\pi\right)$


QPSK SER Versus Phase Noise
0.01
1·10^−3
1·10^−4
1·10^−5
SER
5
6
7
8
9
10
11
12
Eb/ No, dB
1 degree rms
3 degrees rms
5 degrees rms
7 degrees rms

Symbol Error Rates with Timing Errors Present

J.A. Crawford
March 24, 2004

Baseband Synchronization

Transmitter End: Assumed to be square-root raised cosine pulse-train with BT= 0.50

$T := 1$ Assume one symbol per second

$\alpha := 0.5$

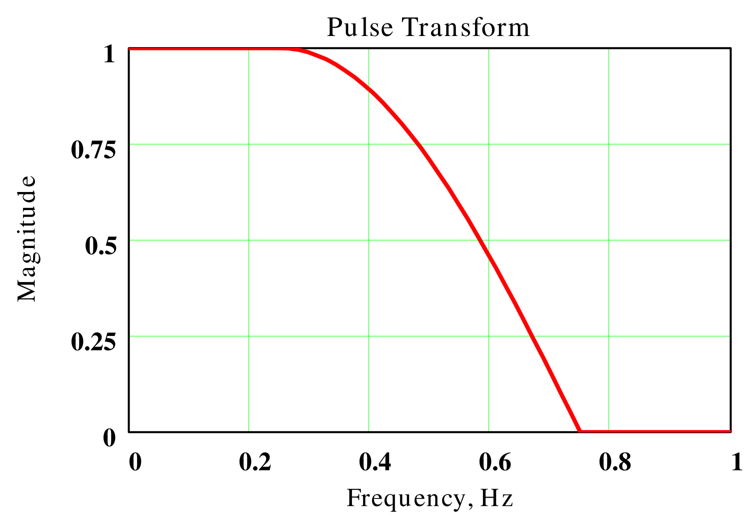
$$H_{srrc}(f) := \left| \begin{array}{l} x \leftarrow \sqrt{T} \ \text{ if } \ |f| < \dfrac{(1-\alpha)}{2 \cdot T} \\ x \leftarrow \sqrt{\dfrac{T}{2} \cdot \left[1 - \sin\left[\dfrac{\pi \cdot T}{\alpha} \cdot \left(f - \dfrac{1}{2 \cdot T} \right) \right] \right]} \ \text{ otherwise} \\ x \leftarrow 0 \ \text{ if } \ |f| > \dfrac{1+\alpha}{2 \cdot T} \\ x \leftarrow x \end{array} \right.$$

$N_{pts} := 1024$ $sps := 32$ Samples per symbol used

$nn := 0 .. N_{pts}$

$fsw_{nn} := \dfrac{sps}{2 \cdot T} \cdot \dfrac{nn}{N_{pts}}$ $df := \dfrac{sps}{T} \cdot \dfrac{1}{N_{pts}}$ $dt := \left(df \cdot N_{pts} \right)^{-1}$

$Pls_{nn} := H_{srrc}\left(fsw_{nn} \right)$



Receive End: N=3 Butterworth Filter

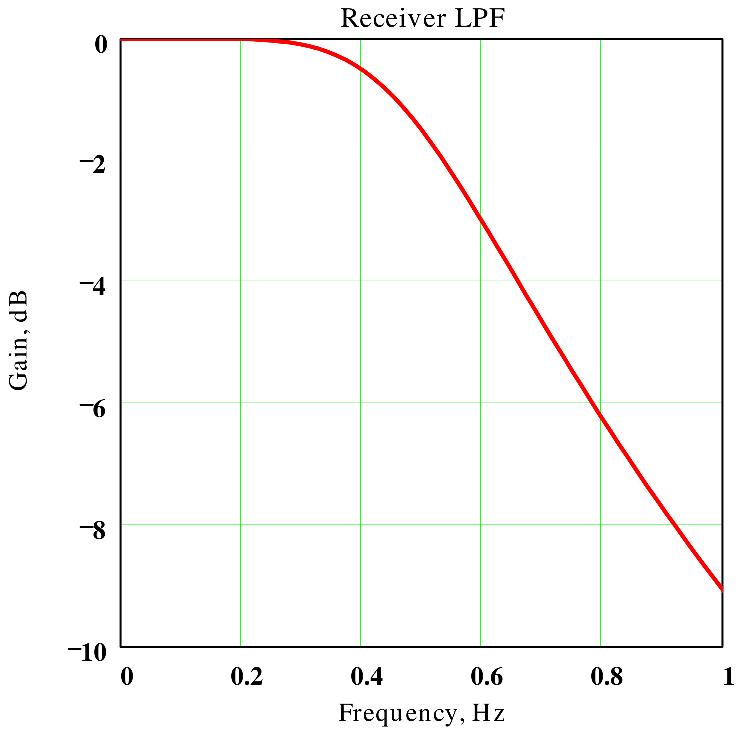
$$jx := \sqrt{-1}$$

$$\omega_c := 2\cdot\pi\cdot 0.50$$

$$H_{rx}(s,\omega_c) := \frac{1}{\left(\frac{s}{\omega_c}\right)^3 + 2\cdot\left(\frac{s}{\omega_c}\right)^2 + 2\cdot\frac{s}{\omega_c} + 1}$$

$$LPF_{nn} := 10\cdot\log\left(\left|H_{rx}(jx\cdot 2\cdot\pi\cdot fsw_{nn},\omega_c)\right|\right)$$

$$NBW := \int_0^{8\cdot T^{-1}} \left(\left|H_{rx}(jx\cdot 2\cdot\pi\cdot ff,\omega_c)\right|\right)^2 dff$$



$NBW = 0.524$

Compute Receive Pulse Shape Out of LPF WITHOUT Noise Present

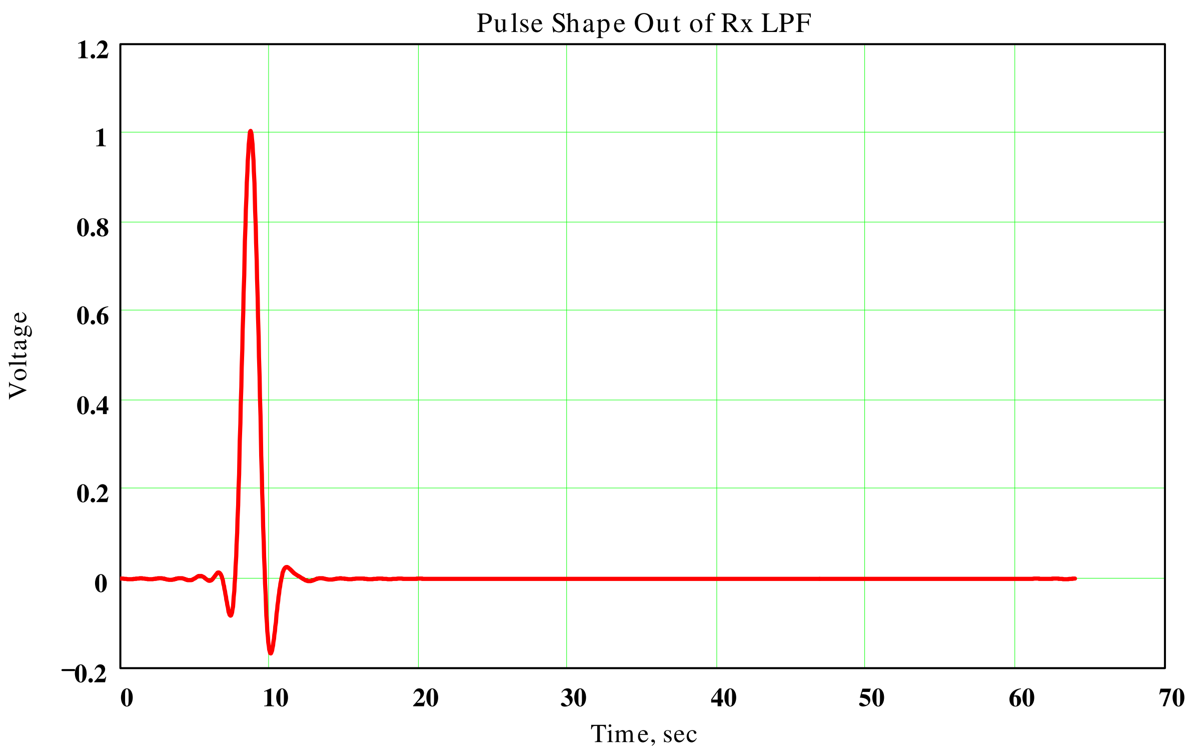
$\tau := 8 \cdot T$

$$RxPls(f, \omega_c) := H_{srrc}(f) \cdot H_{rx}(jx \cdot 2 \cdot \pi \cdot f, \omega_c) \cdot e^{-jx \cdot 2 \cdot \pi \cdot f \cdot \tau}$$

$\omega_c = 3.142$

$Rxp_{nn} := RxPls(fsw_{nn}, \omega_c)$

$rxp := IFFT(Rxp) \cdot \dfrac{dt^{-1}}{2 \cdot N_{pts}}$ $\qquad pp := 0 .. rows(rxp) - 1$



$Pk := max(rxp)$

$PkLoc := \sum_{pp} if(rxp_{pp} = Pk, pp, 0)$ $\qquad PkLoc = 279$

Save receiver output pulse shape for the peak location +/- 8 symbol times

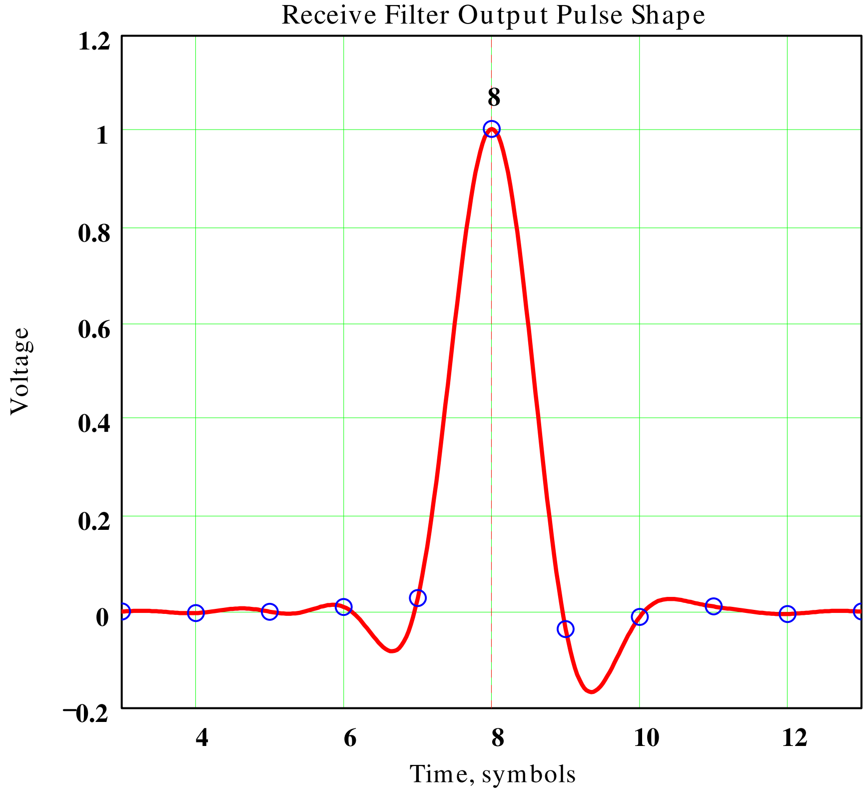
$ii := 0 .. 16 \cdot sps$

$jj := 0 .. 16$

$pulse_{ii} := rxp_{PkLoc-8 \cdot sps+ii}$

$pulse2_{jj} := rxp_{PkLoc-8 \cdot sps+jj \cdot sps}$

$tm_{ii} := (-8 \cdot sps + ii) \cdot dt$



Compute Symbol Error Rate with Noise and Intersymbol Interference

$$C(\omega, t_{err}, L) := \prod_{kk=-L}^{L} \text{if}\left(kk = 0, 1, \cos\left(\omega \cdot \text{linterp}\left(tm, pulse, t_{err} + kk \cdot T\right)\right)\right)$$

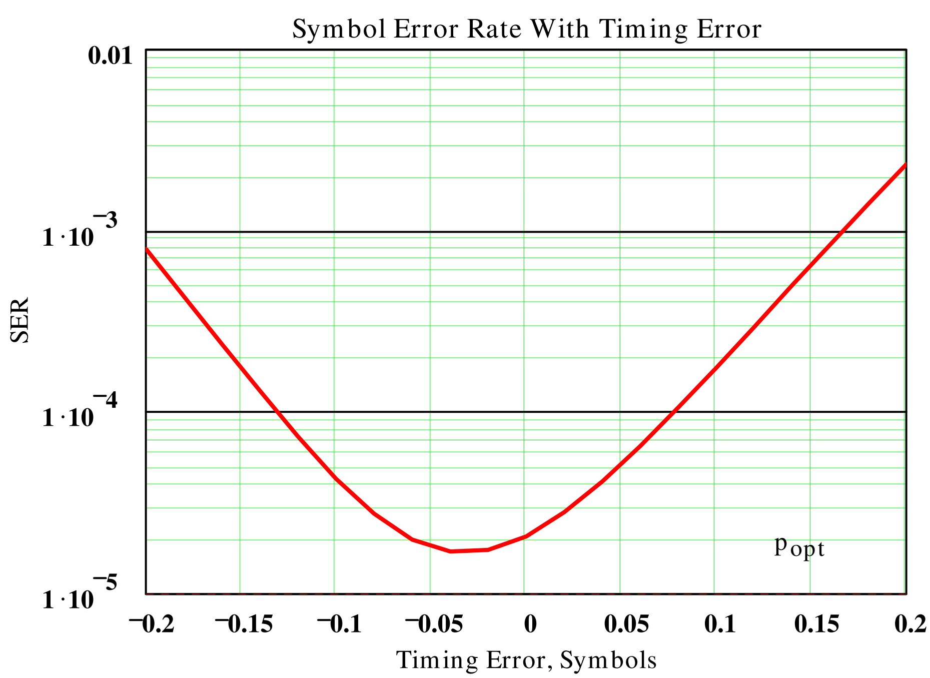
Characteristic Function for the ISI terms

$$Pe(t_{err}, EbNo_{dB}, L) := \begin{vmatrix} N_o \leftarrow 10^{-0.1 \cdot EbNo_{dB}} \\ var \leftarrow N_o \cdot NBW \\ Pe \leftarrow \frac{1}{2} - \int_{0.0000001}^{2 \cdot \pi \cdot \frac{10}{T}} \frac{\sin\left(\omega \cdot \text{linterp}\left(tm, pulse, t_{err}\right)\right)}{\omega} \cdot C(\omega, t_{err}, L) \cdot \frac{e^{\frac{-1}{2} \cdot var \cdot \omega^2}}{\pi} \, d\omega \end{vmatrix}$$

$$Pe(0.0, 9.58, 8) = 2.076 \times 10^{-5}$$

$SNR_{dB} := 9.58$ $\qquad p_{opt} := \frac{1}{2} \cdot \text{erfc}\left(\sqrt{10^{0.1 \cdot SNR_{dB}}}\right)$

$ss := 0 .. 20$ $\qquad terx_{ss} := -0.20 + 0.40 \cdot \frac{ss}{20}$ $\qquad pex_{ss} := Pe\left(terx_{ss}, SNR_{dB}, 8\right)$



$SNR_{dB} = 9.58$

$snr := 10^{0.1 \cdot SNR_{dB}}$

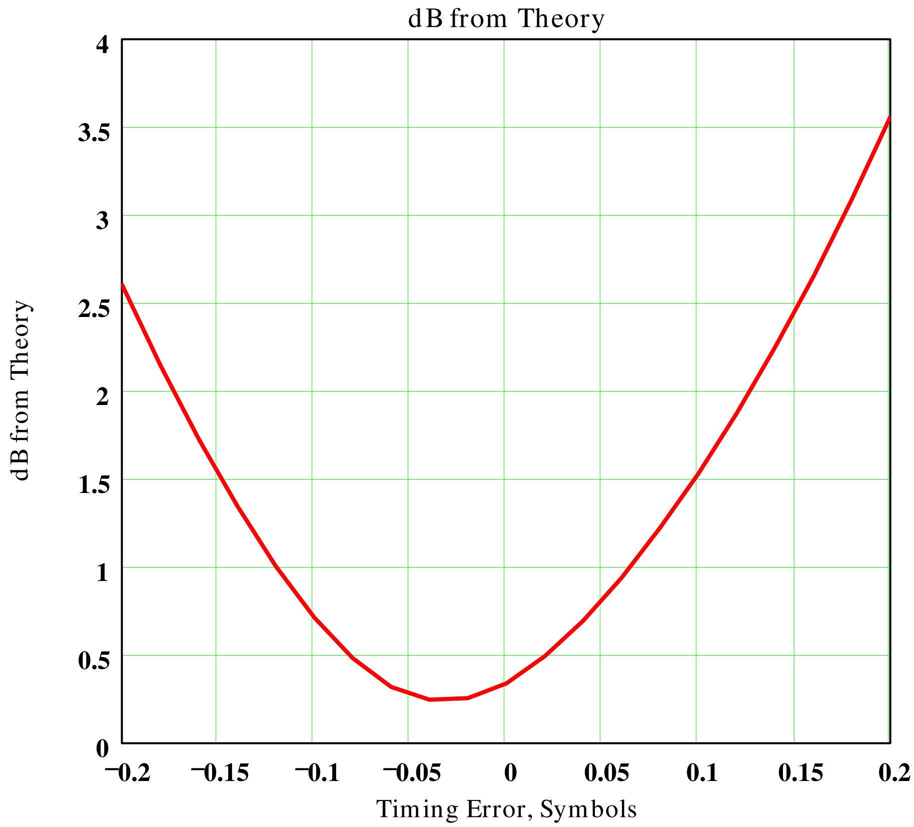
Define inverse erfc() function to assist in computing dB from theory

Given

$$\left(\mathrm{erfc}\left(\sqrt{snr}\right) - pe\right) \cdot 10^{6} = 0$$

inverse_erfc(pe) := Find(snr)

$$deldB_{ss} := SNR_{dB} - 10 \cdot \log\left(\mathrm{inverse\_erfc}\left(2 \cdot pex_{ss}\right)\right)$$



min(deldB) = 0.244

Accumlated Results

$datout^{\langle 0 \rangle} := terx$

$datout^{\langle 1 \rangle} := deldB$

C:\..\BS SER BT0.60.dat

datout

Read in previously computed results

d40 := C:\..\BS SER BT0.40.dat

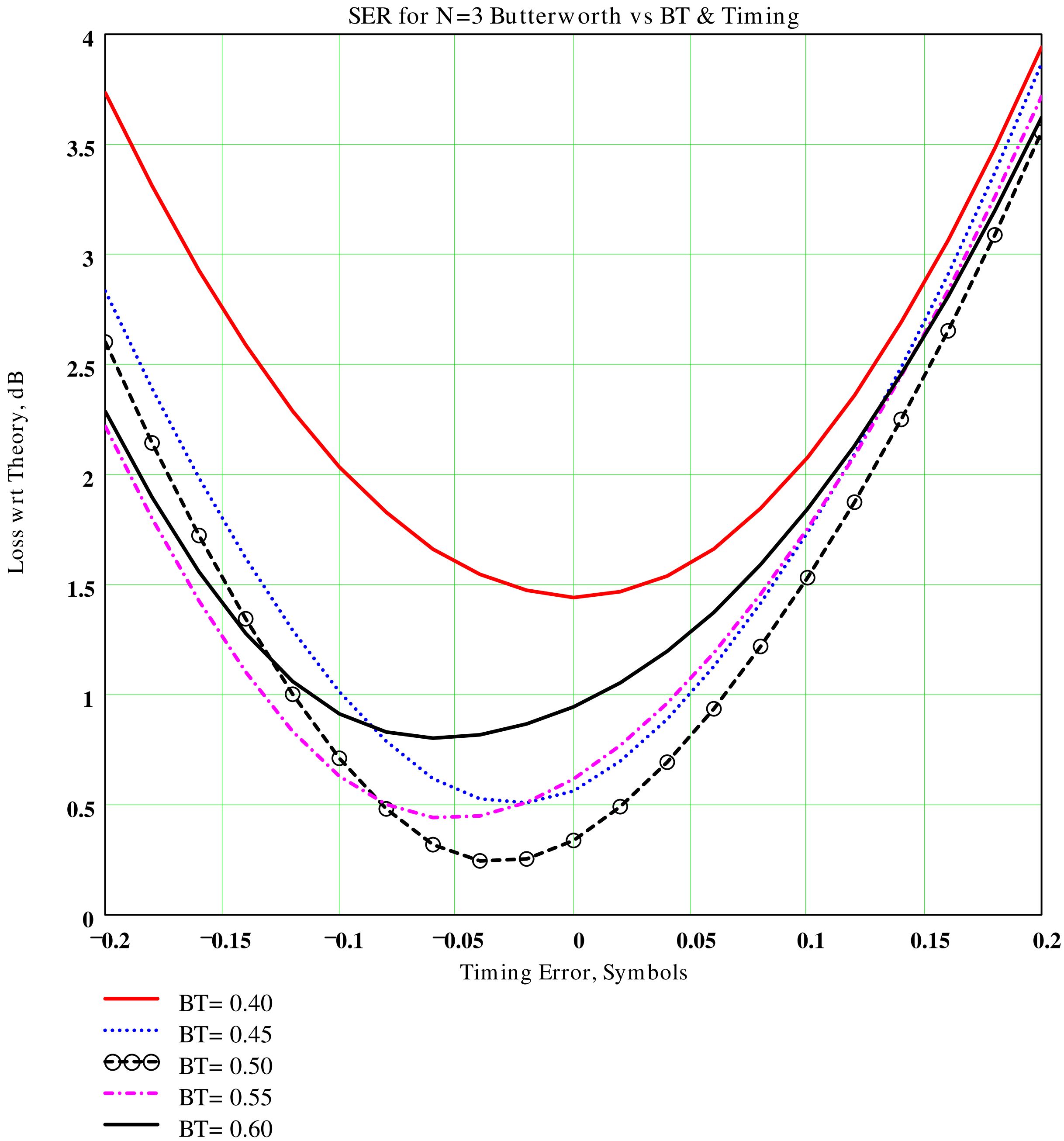
d45 := C:\..\BS SER BT0.45.dat

d50 := C:\..\BS SER BT0.50.dat

d55 := C:\..\BS SER BT0.55.dat

d60 := C:\..\BS SER BT0.60.dat

$$\omega_c \cdot \frac{T}{2 \cdot \pi} = 0.5 \qquad SNR_{dB} = 9.58$$



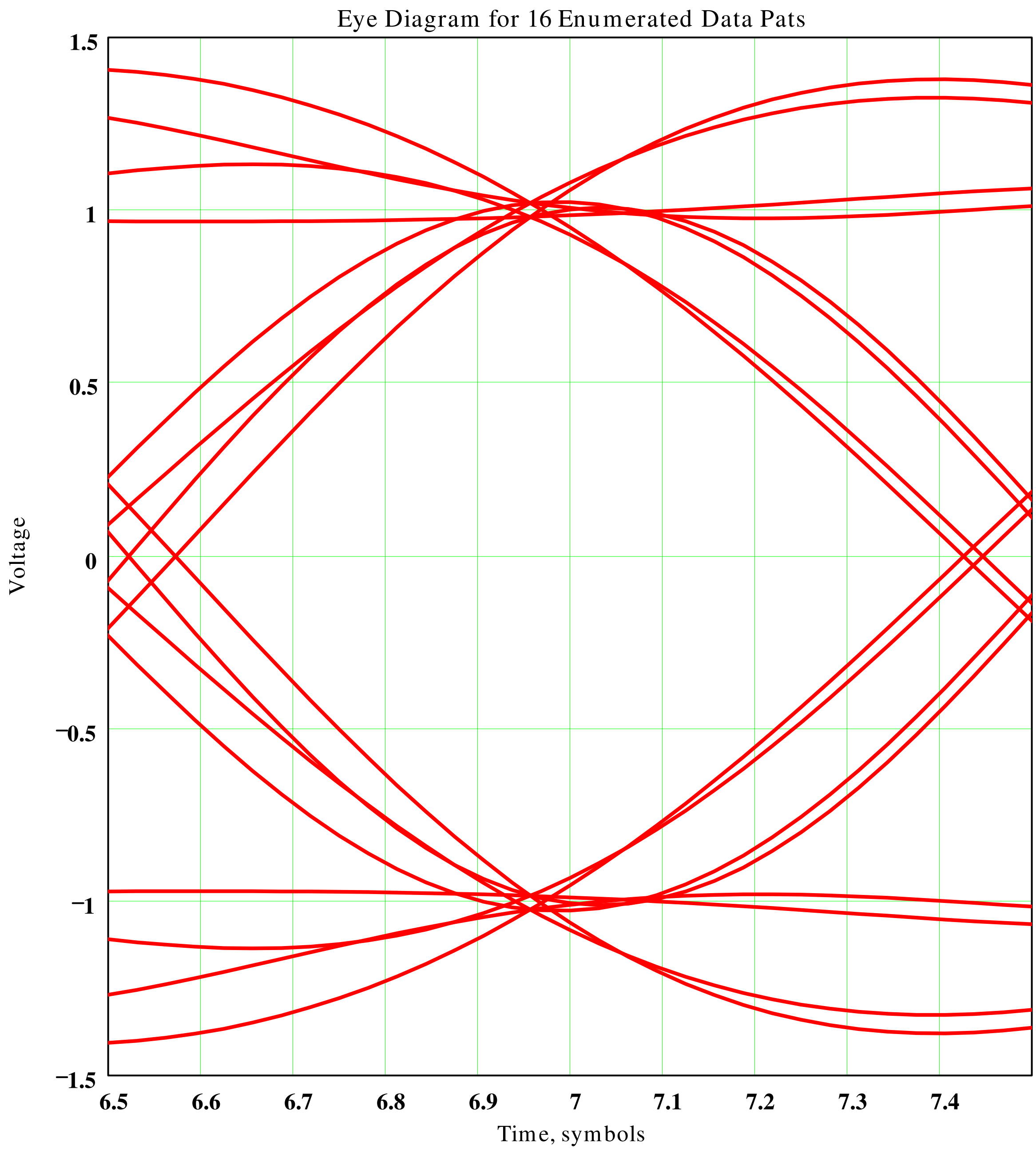
$$pat := \begin{pmatrix} 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{pmatrix}^T$$

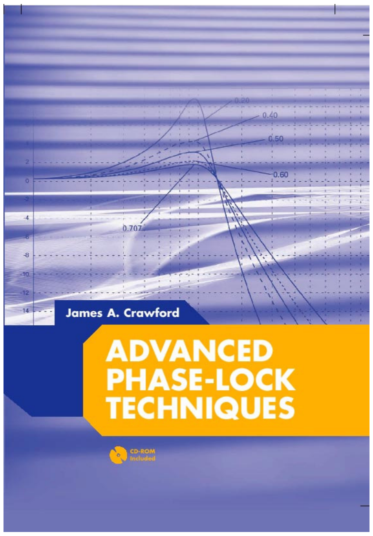
```
pt(datpat) := | for  pp ∈ 0 .. 16·sps
              |   ps_pp ← 0
              | for  ii ∈ 0 .. 3
              |   | dat ← if(pat_datpat,ii = 1, 1, −1)
              |   | for  pp ∈ 0 .. 16·sps
              |   |   ps_pp ← ps_pp + dat·pulse_mod(pp+ii·sps, 16·sps)
              | ps ← ps
```

$pt0 := pt(0)$ $pt3 := pt(3)$ $pt6 := pt(6)$ $pt9 := pt(9)$ $pt12 := pt(12)$ $pt15 := pt(15)$

$pt1 := pt(1)$ $pt4 := pt(4)$ $pt7 := pt(7)$ $pt10 := pt(10)$ $pt13 := pt(13)$

$pt2 := pt(2)$ $pt5 := pt(5)$ $pt8 := pt(8)$ $pt11 := pt(11)$ $pt14 := pt(14)$


Eye Diagram for 16 Enumerated Data Pats
1.5
1
0.5
0
−0.5
−1
−1.5
Voltage
6.5
6.6
6.7
6.8
6.9
7
7.1
7.2
7.3
7.4
Time, symbols



Advanced Phase-Lock Techniques

James A. Crawford

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Artech House

510 pages, 480 figures, 1200 equations
CD-ROM with all MATLAB scripts

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2	*Design Notes* A compilation of design notes and formulas that are developed in details separately in the text. Includes an exhaustive list of closed-form results for the classic type-2 PLL, many of which have not been published before.	44
3	*Fundamental Limits* A detailed discussion of the many fundamental limits that PLL designers may have to be attentive to or else never achieve their lofty performance objectives, e.g., Paley-Wiener Criterion, Poisson Sum, Time-Bandwidth Product.	38
4	*Noise in PLL-Based Systems* An extensive look at noise, its sources, and its modeling in PLL systems. Includes special attention to $1/f$ noise, and the creation of custom noise sources that exhibit specific power spectral densities.	66
5	*System Performance* A detailed look at phase noise and clock-jitter, and their effects on system performance. Attention given to transmitters, receivers, and specific signaling waveforms like OFDM, M-QAM, M-PSK. Relationships between EVM and image suppression are presented for the first time. The effect of phase noise on channel capacity and channel cutoff rate are also developed.	48
6	*Fundamental Concepts for Continuous-Time Systems* A thorough examination of the classical continuous-time PLL up through 4^{th}-order. The powerful Haggai constant phase-margin architecture is presented along with the type-3 PLL. Pseudo-continuous PLL systems (the most common PLL type in use today) are examined rigorously. Transient response calculation methods, 9 in total, are discussed in detail.	71
7	*Fundamental Concepts for Sampled-Data Control Systems* A thorough discussion of sampling effects in continuous-time systems is developed in terms of the z-transform, and closed-form results given through 4^{th}-order.	32
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